Python Programming

Eigen Decomposition & Singular Value Decomposition

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Outlines

- Eigen Decomposition
- Singular Value Decomposition

Spectral/ Eigen Decomposition

Definition

Spectral Decomposition (also called **Eigen Decomposition**) is a mathematical method used to decompose a **square matrix** into its **eigenvalues** and **eigenvectors**. It is primarily applied to **symmetric matrices** and plays a fundamental role in **linear algebra, PCA, and machine learning**.

Spectral/ Eigen Decomposition

Mathematics

For a **symmetric matrix** A, the **spectral decomposition** states that:

$$A = V\Lambda V^T$$

where:

A is a $n \times n$ real symmetric matrix.

V is an **orthogonal matrix** whose columns are **eigenvectors** of A.

Λ is a **diagonal matrix** whose diagonal elements are the **eigenvalues** of A.

Spectral/ Eigen Decomposition

Mathematics

- Since V is orthogonal ($V^T V = I$), we can rewrite:

$$A = V\Lambda V^T$$

$$AV = V\Lambda$$

- This means A **transforms each eigenvector into a scalar multiple of itself (eigenvalue)**.

Spectral/ Eigen Decomposition

Limitation

- A matrix A **has a spectral decomposition if** it is:
  - **Square matrix** $(n \times n)$.
  - **Symmetric matrix** $(A = A^T)$ – ensures real eigenvalues and orthogonal eigenvectors.
  - For a *non-symmetric matrix*, we use **Singular Value Decomposition (SVD)** instead.

Spectral/ Eigen Decomposition

Summary

Spectral decomposition factorizes a symmetric matrix into eigenvectors and eigenvalues.

Used in PCA, quantum mechanics, and linear transformations.

Exists only for **symmetric** matrices (otherwise, use SVD).

Singular Value Decomposition (SVD)

Definition

Singular Value Decomposition (**SVD**) is a powerful **matrix factorization technique** used in **linear algebra, PCA, machine learning, and numerical computing**. It decomposes any $m \times n$ **matrix** A into three simpler matrices.

Singular Value Decomposition (SVD)

Mathematics

- For any real $m \times n$ **matrix** A, the **SVD** is:

$$A = U\Sigma V^T$$

  where $U^T U = I, V^T V = I$

  A $(m \times n)$ is an any **real matrix**.

  U $(m \times m)$ **contains left singular vectors** (eigenvectors of AA^T).

  Σ $(m \times n)$ **is a diagonal matrix** with singular values.

  V $(n \times n)$ **contains right singular vectors** (eigenvectors of $A^T A$).

Singular Value Decomposition (SVD)

Mathematics

where:

$$U^T U = I$$
$$V^T V = I$$

U **and** V **are orthogonal** $\rightarrow$ their columns form an **orthonormal basis**.

Σ **contains singular values** (which determine how much variance is along each principal axis).

Singular Value Decomposition (SVD)

The meaning of U, Σ, and V

Columns of U: Eigenvectors of AA^T (span the row space of A).

Columns of V: Eigenvectors of A^TA (span the column space of A).

Diagonal values of Σ: Square roots of eigenvalues of A^TA or AA^T.

- If A has **rank** r:

  The first r singular values are **nonzero**.

  The last $(m - r)$ or $(n - r)$ singular values are **zero**.

Singular Value Decomposition (SVD)

Mathematical derivation of A^TA

Why V contains the eigenvectors of A^TA?

$\because apply\ SVD$

$A^TA = (U\Sigma V^T)^T(U\Sigma V^T)$

$\because transpose\ property$

$A^TA = V\Sigma^T U^T \cdot U\Sigma V^T$ Since U is orthogonal ($U^TU = I$)

$A^TA = V\Sigma^T\Sigma V^T = V\Sigma^2 V^T$ Since $\Sigma^T\Sigma$ is a diagonal matrix of squared singular values (please refer to Eigen Decomposition)

$A^TAV = V\Sigma^2$

$\therefore$ V is the eigenvector of A^TA with eigenvalues given by $\Sigma^T\Sigma$

Singular Value Decomposition (SVD)

Mathematical derivation of AA^T

Why U contains the eigenvectors of AA^T?

$\because apply\ SVD$

$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T$

$\because transpose\ property$

$AA^T = U\Sigma V^T \cdot V\Sigma^T U^T$ Since V is orthogonal ($V^T V = I$)

$AA^T = U\Sigma^T\Sigma U^T = U\Sigma^2 U^T$ Since $\Sigma^T\Sigma$ is a diagonal matrix of squared singular values (please refer to Eigen Decomposition)

$AA^T U = U\Sigma^2$

$\therefore U$ is the eigenvector of AA^T with eigenvalues given by $\Sigma^T\Sigma$

The End

Thank you for your attention!

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